QUANTUM COHOMOLOGY AND S^1-ACTIONS WITH ISOLATED FIXED POINTS

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Abstract. This paper studies symplectic manifolds that admit semi-free circle actions with isolated fixed points. We prove, using results on the Seidel element [4], that the (small) quantum cohomology of a $2n$ dimensional manifold of this type is isomorphic to the (small) quantum cohomology of a product of n copies of $\mathbb{P}^1$. This generalizes a result due to Tolman and Witsman [11].

1. Introduction

Let (M,ω) be a $2n$ dimensional compact, connected, symplectic manifold, and let $\{\lambda_t\} = \lambda : S^1 \longrightarrow \mathrm{Symp}(M,\omega)$ be a symplectic circle action on M, that is, if X is the vector field generating the action, then $\mathcal{L}_X\omega = d\iota_X\omega = 0$. Recall that the action is semi-free if it is free on $M\backslash M^{S^1}$. This is equivalent to say that the only *weights* at every fixed point are ± 1. A circle action is said to be Hamiltonian if there is a C^∞ function $H : M \longrightarrow \mathbb{R}$ such that $\iota_X\omega = -dH$. Such a function is called a Hamiltonian for the action.

Tolman and Weitsman proved in [11] that if the action is semi-free and admits only isolated fixed points, then the action must be Hamiltonian provided that there is at least one fixed point. There is a great deal of information concerning the topology of manifolds carrying such actions. The first result in this direction is due to Hattori [2]. He proves that there is an isomorphism from the cohomology ring $H^*(M;\mathbb{Z})$ to the cohomology ring of a product of n copies of $\mathbb{P}^1$. Moreover, this isomorphism preserves Chern classes. In [11] Tolman and Weitsman generalize Hattori's result to equivariant cohomology. The main result of this paper is to extend this result to quantum cohomology. In §3.1 we prove that M is almost Fano manifold, therefore we can use polynomial coefficients $\Lambda := \mathbb{Q}[q_1,\dots,q_n]$ for the quantum cohomology ring. The main theorem is the following.

Theorem 1.1. *Let (M,ω) be a $2n$-dimensional compact connected symplectic manifold. Assume M admits a semi-free circle action with a finite non-empty set of fixed points. Then there is an isomorphism of (small) quantum cohomology*

$$QH^*(M;\Lambda) \cong QH^*((\mathbb{P}^1)^n;\Lambda).$$

Note that we can compute directly the quantum cohomology of $\mathbb{P}^1\times\cdots\times\mathbb{P}^1$ to get the following result.

Corollary 1.2. *The (small) quantum cohomology of M is given by*

$$QH^*(M;\Lambda) \cong QH^*((\mathbb{P}^1)^n;\Lambda) \cong \frac{\mathbb{Q}[x_1,\dots,x_n,q_1,\dots,q_n]}{< x_i * x_i - q_i >}$$

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where $\deg(x_i) = 2$ *and* $\deg q_i = 4$.

Moreover, all other products are given by

$$x_{i_1} * \cdots * x_{i_k} = x_{i_1} \smile \cdots \smile x_{i_k}$$

for $i_1 < \cdots < i_k$. *Here the product on the left is the quantum product, while the term on the right is the usual cup product.*

To prove Theorem 1.1 we will construct a set of generators $\{x_i\}$ of the cohomology ring $H^*(M;\mathbb{Z})$. Then we prove in Lemma 4.1 that the quantum products of these generators satisfy the expected relations given in Corollary 1.2.

To get this relations we use a result of McDuff-Tolman [4] to understand how the Seidel automorphism acts on the generators. We will see in Corollary 3.13 that this action do not have higher order terms, that is the automorphism is given by single homogeneous terms in quantum cohomology. Thus the Seidel automorphism acts by permutation of the elements in the basis. To construct such generators for the cohomology ring, we will adopt the tools that Tolman and Weitsman developed to prove the following theorem.

Theorem 1.3 ([11])**.** *Let* (M, ω) *be a compact, connected symplectic manifold with a semi-free, Hamiltonian circle action with isolated fixed points. Then, there is an isomorphism of rings* $H^*_{S^1}(M) \simeq H^*_{S^1}(((\mathbb{P}^1)^n))$ *which takes the equivariant Chern classes of* M *to those of* $(\mathbb{P}^1)^n$. *Therefore the equivariant cohomology ring is given by*

$$H^*_{S^1}(M) = \mathbb{Z}[a_1, \ldots, a_n, y]/(a_i y - a_i^2).$$

Here $a_i \in H^2_{S^1}(M)$ *and the equivariant Chern series is given by* $c_t(M) = \sum_i c_i(M)t^i$ *where*

$$c_t(M) = \prod_i (1 + t(2a_i - y)).$$

Although Tolman and Weitsman use equivariant cohomology for getting an invariant base for $H^*(M;\mathbb{Z})$, the results of McDuff-Tolman require a more geometric description of the basis. Therefore the crucial element in most of the results of this paper is having geometric representatives of the cycles dual to the cohomology basis. These geometric representatives are defined by the Morse complex of the Hamiltonian function.

The paper is organized as follows. All the Morse theoretical constructions are in §2.1. In section 2.2 we use equivariant cohomology to provide an invariant basis for cohomology. Then we establish the relation with the Morse cycles. In §3.1 we define the quantum cohomology ring and we get results that help to reduce the quantum product formulas. In §3.3 we define the Seidel automorphism in quantum cohomology. In §3.4 we relate the Seidel automorphism with invariant chains. Then we compute explicitly the Seidel element. Finally in §4 we use the associativity of the quantum product together with some dimensional arguments to provide the proof of Theorem 1.1.

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2. Morse Theory and Equivariant Cohomology

In this section we establish all the tools we need to prove Theorem 1.1. We start in §2.1 with basic definitions of Morse theory. For more details the reader can consult [1, 8] for Morse theory.

Following the approach of [4], we will construct invariant Morse cycles to be able to calculate the Seidel element of M. This will be done in the next section. We introduce equivariant cohomology to identify a basis in cohomology and describe the relation with Morse cycles. At the end, we provide several results that will be necessary in §4.

2.1. **Morse Theory.**

As in §1, let (M, ω) be a symplectic $2n$-dimensional manifold with S^1 action generated by a Hamiltonian function H. Thus $\iota_X \omega = -dH$ and $X = J\mathrm{grad}(H)$, where the gradient is taken respect to the metric $g_J(x, y) = \omega(x, Jy)$ for an ω-compatible S^1-invariant almost complex structure J. With respect to this metric, H is a (perfect) Morse function [3] and the zeroes of X are exactly the critical points of H. For each fixed point $p \in M^{S^1}$, denote by $\alpha(p)$ the index of p and let $m(p)$ be the sum of weights at p. Since the action is semi-free $m(p) = n_+(p) - n_-(p)$ where $n_+(p)$ is the number of positive weights and $n_-(p)$ the number of negative ones. Then $\alpha(p) = 2n_-(p) = n - m(p)$.

In order to understand the (co)homology of M in terms of S^1-invariant cycles, we will consider the *stable* and *unstable* manifolds with respect to the gradient flow $-\mathrm{grad}(H)$. More precisely, let p, q be critical points of H. Define the stable and unstable manifolds by

$$W^s(q) = \{\gamma : \mathbb{R} \longrightarrow M | \lim_{t \longrightarrow \infty} \gamma(t) = q\},$$
$$W^u(p) = \{\gamma : \mathbb{R} \longrightarrow M | \lim_{t \longrightarrow -\infty} \gamma(t) = p\}.$$

Here $\gamma(t)$ satisfies the gradient flow equation

$$\gamma'(t) = -\mathrm{grad}H(\gamma(t)).$$

These spaces are manifolds of dimension

$$\dim W^s(q) = 2n - \alpha(q) \quad \text{and} \quad \dim W^u(p) = \alpha(p),$$

and the evaluation map $\gamma \mapsto \gamma(0)$ induces smooth embeddings into M

$$E_q : W^s(q) \longrightarrow M \quad \text{and} \quad E_p : W^u(p) \longrightarrow M.$$

When these manifolds intersect transversally for all fixed points p, q, the gradient flow is said to be Morse-Smale [1, 8]. Under this circumstance we say that the pair (H, g_J) is *Morse regular*.

In [8] Schwartz proved that there is a way of *partially compactifying* these manifolds and that there are natural extensions of the evaluation maps so that these compactifications with their evaluation maps $E_p : \overline{W^s(p)} \longrightarrow M$ and $E_q : \overline{W^u(q)} \longrightarrow M$, define *pseudocycles*. The compactification of $W^s(p)$ is made by adding *broken trajectories* through fixed points of index $\alpha(p) - 1$. When the action is semi-free and admits isolated fixed points, all the fixed points have even index (see comment after Theorem 2.2), therefore $W^s(p)$ is already compact in the sense of Schwartz. Thus $W^s(p)$ is itself a pseudocycle. The same is true for $W^s(x)$. It is well known that pseudocycles define classes in homology (see [6]). We will denote

by $[W^u(p)] \in H_{\alpha(x)}(M;\mathbb{Z})$ and $[W^s(p)] \in H_{n-\alpha(x)}(M;\mathbb{Z})$ the homology classes defined by these manifolds. To make these classes S^1-invariant we need to consider a special type of almost complex structure, as we explain below.

Assume (M,ω) admits a Hamiltonian S^1-action with isolated fixed points. Each fixed point $p \in M$ has a neighborhood $U(p)$ that is diffeomorphic to a neighborhood of zero in a $2n$-dimensional Hermitian vector space $E(p) = E_1 \oplus \cdots \oplus E_n$, in such a way that the moment map H is given by

$$H(v_1,\ldots,v_n) = \sum_j \pi m_j |v_j|^2$$

and S^1 acts in E_j just by multiplication by $e^{2\pi i m_j}$. Here the numbers $m_j \in \mathbb{Z}$ are exactly the weights of the action. Under the identification above, the almost-complex structure J is the standard complex structure on the Hermitian vector space $E(p)$. Observe that $E(p)$ can be written as $E^+ \oplus E^-$ where $E^{\pm}$ is the sum of the E_j where $m_j > 0$ or $m_j < 0$ respectively. We can call the spaces $E^{\pm}$ the positive and negative normal bundles to the point p.

If we start with any compatible almost complex structure J near the fixed points, we can extend J to an S^1-invariant ω-compatible almost complex structure J_M on M whose restriction to the open sets $U(p)$ is J. Denote by $\mathcal{J}_{\text{inv}}(M)$ the set of all J that are equal to J_M near the fixed points.

The following lemma shows that it is possible to acquire regularity with generic almost-complex structures.

Lemma 2.1 ([4]). *Suppose that H generates a semi free S^1-action on (M,ω). Then for a generic choice of $J \in \mathcal{J}_{\text{inv}}(M)$ the pair (H, g_J) is Morse regular.*

For the rest of this paper, we will only consider Morse regular pairs (H, g_J) as in the previous lemma.

2.2. **Equivariant Cohomology.** We can start with a quick review of equivariant cohomology. Let ES^1 be a contractible space where S^1 acts freely, and denote $BS^1 = ES^1/S^1$. Then $H^*(BS^1;\mathbb{Z})$ is the polynomial ring $\mathbb{Z}[y]$ where $y \in H^2(BS^1;\mathbb{Z})$.

Let S^1 act on a manifold M. The equivariant cohomology of M, denoted by $H^*_{S^1}(M)$ is defined by $H^*(M \times_{S^1} ES^1;\mathbb{Z})$. Note that $H^*(BS^1;\mathbb{Z})$ is naturally isomorphic to $H^*_{S^1}(pt)$, if $pt \in M$ is a point. Under this construction, we have two natural maps, the projection $p : M \times_{S^1} ES^1 \longrightarrow BS^1$ and the inclusion (as fiber) $i : M \longrightarrow M \times_{S^1} ES^1$. The pullback $p^* : H^*(BS^1;\mathbb{Z}) \longrightarrow H^*_{S^1}(M)$ makes $H^*_{S^1}(M)$ a $H^*(BS^1;\mathbb{Z})$ module, while the restriction $i^* : H^*_{S^1}(M) \longrightarrow H^*(M)$ is the "reduction" of invariant data to ordinary data. An immediate consequence is that $i^*(y) = 0$.

Let $j : M^{S^1} \longrightarrow M$ be the natural inclusion. In [3] Kirwan proved that if the action is Hamiltonian, the induced map $j^* : H^*_{S^1}(M) \longrightarrow H^*_{S^1}(M^{S^1})$ is injective. The proof of this theorem is based on the following result, where we weaken the statement to match our needs. For a fixed point $p \in M^{S^1}$ we denote by $a|_p := (j_p)^*(a)$ where $(j_p)^* : H^*_{S^1}(M) \longrightarrow H^*_{S^1}(p)$ and j_p is the obvious inclusion.

Theorem 2.2 ([3]). *Let the circle act on a symplectic manifold M in a Hamiltonian way. Assume the action is semi-free and that there are only isolated fixed points. Let $p \in M$ be a fixed point of index $2k$. Then there exists a unique class $a_p \in H^{2k}_{S^1}(M)$*

such that $a_p|_p = (-1)^k y^k$, and $a_p|_{p'} = 0$ for all other fixed points p' of index less than or equal to $2k$. Moreover, if we consider all fixed points, the classes a_p form a basis for $H^*_{S^1}(M)$ as a $H^*(BS^1;\mathbb{Z})$ module.

As a remark on the previous theorem, note that the term $(-1)^k y^k$ is the equivariant Euler class of the negative normal bundle at p.

As stated in §1, there is an isomorphism $H_*(M;\mathbb{Z}) \cong H_*(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1;\mathbb{Z})$ if M satisfy the hypothesis of Theorem 2.2. Since H is perfect there are exactly $\dim(H_{2k}(M)) = \binom{n}{k}$ critical points of index $2k$. In [2, 11], the above isomorphism is proved by counting fixed points. We will not discuss the proof here.

Denote the points of index 2 by $p_1, \ldots, p_n$. In the light of Theorem 2.2 for each fixed point we get classes $a_1, \ldots, a_n \in H^2_{S^1}(M)$ such that

$$\begin{aligned} a_j|_{p_j} &= -y \\ a_j|_p &= 0 \qquad \text{for all other fixed points } p \text{ of index 0 or 1.} \end{aligned} \tag{1}$$

These classes satisfy the following Proposition.

Proposition 2.3 ([11, Prop 4.4]). *Let I be a subset of $\{1, \ldots, n\}$ with k elements. There exist a unique fixed point p_I of index $2k$ such that*

$$a_j|_{p_I} = -y \quad \textit{if and only if} \quad j \in I$$

and $a_j|_{p_I} = 0$ otherwise.

Proposition 2.3 identifies the fixed points in M with subsets I of $\mathcal{S} := \{1, \ldots, n\}$. Observe that the cohomology class $a_I := \prod_{i\in I} a_i \in H^{2k}_{S^1}(M)$ is the same as the class a_{p_I} mentioned in Theorem 2.2. Moreover this class is such that

$$a_I|_{p_J} = (-1)^k y^k \text{ if and only if } I \subseteq J \tag{2}$$

and it is zero otherwise.

Remark 2.4. *The class a_0, associated to the unique point of index zero, takes the value $1 \in H^0_{S^1}(pt)$ when restricted to any fixed point. Therefore it is the identity element in the ring $H^*_{S^1}(M)$. Denote ya_0 by y.*

If we apply the same results to the Hamiltonian function $-H$, we obtain unique classes $b_J \in H^{2n-2k}_{S^1}(M)$ associated to each p_J of index $2k$ such that $b_J|_{p_J} = (-1)^{n-k} y^{n-k}$ and is zero when restricted to all other fixed points of index greater or equal to $2k$. These classes also form a basis of $H^*_{S^1}(M)$. The next proposition establishes the relation with the former basis.

Proposition 2.5. *Let $I = \{i_1, \ldots, i_k\}$ and let $I^c = \{i_{k+1}, \ldots, i_n\}$ be its complement. Then the classes b_I satisfy the following relation*

$$b_I = \sigma_{n-k} + y\sigma_{n-k-1} + \cdots + y^{n-k}, \tag{3}$$

where σ_i is the i-th symmetric function in the variables $a_{i_{k+1}}, \ldots, a_{i_n}$.

Proof. There are two ways of seeing this. One is just by checking that when we restrict the right side of Equation (3) to the fixed point p_J we get by means of (2) the same as $b_I|_{p_J}$. We can also check by hand, i.e. by restriction to all fixed points. Once again, applying (2) we get

$$b_{\{i\}^c} = a_i + y,$$

Finally we can check that

$$b_I|_{p_J} = \{\prod_{i\notin I}(a_i + y)\}|_{p_J}.$$

for all fixed points p_J.

$\square$

Consider a point p_I of index $2k$ and associate the class $a_I \in H^{2k}_{S^1}(M)$ as before. When we restrict a_I to M we obtain a class $a_I|_M \in H^{2k}(M;\mathbb{Z})$. By taking the Poincaré dual of $a_I|_M$, we get a homology class $p_I^+ \in H_{2n-2k}(M;\mathbb{Z})$. Similarly using the class b_I we get a homology class $p_I^- \in H_{2k}(M;\mathbb{Z})$. Here is an immediate corollary of Proposition 2.5.

Corollary 2.6. *The class p_I^- is the same as the class $p_{I^c}^+$.*

Proof. This is clear because the variable y is mapped to zero under reduction to usual cohomology. Now use that $\sigma_{n-k} = a_{I^c}$.

$\square$

The last part of this section establishes the relation of the $p_I^{\pm}$ classes with the stable and unstable manifolds of §2.1. This is summarized in the following proposition. Remember that we are working with an almost-complex structure J in $\mathcal{J}_{\text{inv}}(M)$. This result would fail without this hypothesis.

Proposition 2.7. *Let p_I be a fixed point of index $2k$. Then the classes p_I^- and p_I^+ are exactly the same as the classes $[W^u(p_I)]$ and $[W^s(p_I)]$ respectively.*

Proof. Recall that ES^1 can be taken to be the infinite dimensional sphere S^∞. Consider a finite dimensional approximation $M^N := M\times_{S^1} S^{2N+1}$ of $M\times_{S^1} ES^1 = M\times_{S^1} S^\infty$ for $N\in\mathbb{N}$ big enough . These are finite dimensional smooth compact manifolds. Since $W^s(p_I)$ is S^1-invariant, there is a natural extension $W^{N,s}(p_I) := W^s(p_I)\times_{S^1} S^{2N+1}$ of $W^s(p_I)$ to M^N. Let X^N be the Poincaré dual of $W^{N,s}(p_I)$ in M^N.

For all N, there is a natural inclusion (as fibre) $i_N : M \hookrightarrow M^N$. Since the inclusions are natural, the restriction $X^N|_M := (i_N)^*(X^N) \in H^*(M)$ is the same as the Poincaré dual of $[W^s(p_I)]$ in M.

Observe that the natural inclusions

$$M^N \hookrightarrow M^{N+1} \hookrightarrow \cdots \lim_N M^N = M\times_{S^1} ES^1$$

induce a sequence

$$\cdots \longrightarrow X^{N+2} \longrightarrow X^{N+1} \longrightarrow X^N$$

given by the restrictions. Thus, by considering the directed limit, there is an element

$$X := \lim_N X^N \in H^*(M\times_{S^1} ES^1) = H^*_{S^1}(M)$$

that restricts to X^N for all N. Naturally, if $i : M \hookrightarrow M\times_{S^1} ES^1$ is the inclusion, then $X|_M := i^*(X) = \mathrm{PD}([W^s(p_I)])$. We claim that X satisfies the same properties as the class a_I, that is, $X|_{p_I} = (-1)^k y^k$ and $X|_p = 0$ for all other fixed points p such that $\alpha(p) \le 2k$. Therefore, by Theorem 2.2 we must have $X = a_I$. Then $\mathrm{PD}(X|_M) = \mathrm{PD}(a_I|_M)$ and the result will follow immediately.

Take a neighborhood $U(p_I)$ around p_I as in §2.1. Thus, $U(p_I)$ is isomorphic to an open neighborhood V of zero in $E^+\oplus E^-$. It is clear that if $U(p_I)$ is small enough, $W^s(p_I)\cap U(p_I)$ is diffeomorphic to $E^+\cap V$. Therefore, the normal bundle

of $W^s(p_I)$ can locally be identified with E^-. Finally, by carrying this localization to X^N and considering the limit, we have $X|_{p_I} = e(E^-) = (-1)^k y^k$, where $e(E^-)$ is the equivariant Euler class of E^-.

To finish the proof, observe that if p is any other fixed point with index less than or equal to $2k$, there is no gradient line from p to p_I. This is because the gradient flow is Morse-Smale. Hence, by using the localization again we obtain that $X|_p = 0$. This proves the proposition.

□

Corollary 2.8. *By the definition of the classes a_I and b_I, we have*

$$[W^u(p_I)] = p_I^- = \mathrm{PD}(b_I|_M) \text{ and } [W^s(p_I)] = p_I^+ = \mathrm{PD}(a_I|_M),$$

therefore the product $[W^u(p_I)] \cap [W^s(p_J)]$ is given by

$$[W^u(p_I)] \cap [W^s(p_J)] = \mathrm{PD}(b_I|_M) \cap \mathrm{PD}(a_J|_M) = \mathrm{PD}(b_I a_J|_M).$$

Corollary 2.9. *By Corollary 2.6 and Proposition 2.7 above we have the "duality" relation $[W^u(p_I)] = [W^s(p_{I^c})]$.*

Remark 2.10. *Let $x_i := a_i|_M \in H^2(M;\mathbb{Z})$. The theory of this section proves that the elements x_i generate the algebra $H^*(M;\mathbb{Z})$. Therefore a basis for the vector space $H^{2k}(M;\mathbb{Z})$ consists of the elements $x_I = x_{i_1} \dots x_{i_k}$ for sets $I = \{i_1 < i_2 \dots < i_k\}$. Moreover, by Theorem 1.3 the first Chern class of M is given by $c_1(M) = 2(x_1 + \dots + x_n)$.*

Proposition 2.7 also provides some information about the existence of gradient lines. More precisely we have the next proposition.

Proposition 2.11. *Let $I = \{i_1, \dots, i_k\} \subset \mathcal{S}$. Take $i_{k+1} \notin I$ and consider $I' = I \cup \{i_{k+1}\}$. Let $A_I := \sum_{i \in I} p_i^- \in H_2(M)$. Then,*

a) *There is a gradient line from $p_{I'}$ to p_I. Moreover, the homology class of the sphere generated by rotating the gradient line by the S^1 action is $p^-_{i_{k+1}}$.*
b) *There is a broken gradient line from $p_{\mathcal{S}}$ to p_I. The class A_{I^c} is then represented by rotating this broken line. Also, $\omega(A_{I^c}) = H_{\max} - H(p_I)$ and $c_1(A_{I^c}) = n + m(p_I)$.*

Proof. To prove there is a gradient line from $p_{I'}$ to p_I we need to show that the intersection $W^u(p_{I'}) \cap W^s(p_I)$ is non-empty. By definition of the intersection product in terms of pseudocycles [6] it is enough to prove that the intersection product of the classes $[W^u(p_{I'})]$ and $[W^s(p_I)]$ is non-zero.

Consider the equivariant cohomology classes $b_{I'}$ and a_I. By Proposition 2.5 we get

$$b_{I'} a_I = a_{I'^c} a_I + yd$$

where $d \in H^*_{S^1}(M)$. Since ${I'}^c = I^c \cap \{i_{k+1}\}^c = \{i_{k+1}\}^c$,

$$a_{I'^c} a_I = a_{\{i_{k+1}\}^c}.$$

Once again by Proposition 2.5

$$a_{\{i_{k+1}\}^c}|_M = b_{i_{k+1}}|_M,$$

thus

$$b_{I'} a_I|_M = b_{i_{k+1}}|_M.$$

Now, using Corollary 2.8 we get

$$[W^u(p_{I'})] \cap [W^s(p_I)] = \mathrm{PD}(b_{I'}a_I|_M) = \mathrm{PD}(b_{i_{k+1}}|_M) = p^-_{i_{k+1}} \neq 0. \tag{4}$$

Therefore, there is a gradient line, thus a whole *gradient sphere* A, just by rotating the gradient line. Note that there can be more than one gradient sphere from $p_{I'}$ to p_I. We claim that all these gradient spheres must be homologous.

It is not hard to see from the construction of A that

$$\omega(A) = \int_A \omega = H(p_{I'}) - H(p_I).$$

Therefore if A' is another gradient sphere joining $p_{I'}$ and p_I, $\omega(A) = \omega(A')$. Also observe that if ω' is any S^1-invariant form sufficiently close to ω then $\omega(A) = \omega'(A)$. Now since the symplectic condition is an open condition we can perturb ω to obtain a new symplectic form ω' close to ω. By averaging respect to the group action, we can assume the form ω' to be S^1-invariant. This proves that the classes A' and A have the same symplectic area, that is $\omega'(A) = \omega'(A')$, for an open set of symplectic forms ω'. Since M is simply connected and there is no torsion A must be homologous to A'. Finally by Equation (4) this sphere must be in class $p^-_{i_{k+1}}$.

To prove the second part, we can do the same process for each point in $I^c = \{i_{k+1} \dots i_n\}$. Then getting a sequence of gradient lines

$$p_{\mathcal{S}} \xrightarrow{\gamma_1} p_{\mathcal{S}-\{i_{n-1}\}} \cdots p_{I\cup\{i_{k+1}\}} \xrightarrow{\gamma_{n-k}} p_I.$$

It is clear now that the chain of gradient spheres obtained by rotating this broken gradient line must be in class A_{I^c}. Note that we could also use a gluing argument as in [9] to prove that there is an honest gradient line from $p_{\mathcal{S}}$ to p_I. Thus $\omega(A_{I^c}) = H_{\max} - H(p_I)$ and $c_1(A_{I^c}) = m(p_I) - m(p_{\mathcal{S}}) = n + m(p_I)$.

□

3. Quantum Cohomology and the Seidel Automorphism

3.1. **Small Quantum Cohomology.** In the literature, there are several definitions of quantum cohomology. In this section we make precise the definition of the quantum cohomology we are using, assuming the definition of genus zero Gromov-Witten invariants. We will follow entirely the approach of [6, Chapter 11].

Let Λ_ω be the usual *Novikov ring* of (M, ω). We recall that Λ_ω is the completion of the group ring of $H_2(M) := H_2(M;\mathbb{Z})/\text{Torsion}$. It consists of all (possibly infinite) formal sums of the form

$$\lambda = \sum_{A \in H_2(M)} \lambda_A e^A$$

where $\lambda_A \in \mathbb{R}$ and the sum satisfies the finiteness condition

$$\#\{A \in H_2(M) | \lambda_A \neq 0, \omega(A) \leq c\} < \infty$$

for every real number c. By definition, $\deg(e^A) = 2c_1(A)$, where c_1 is the first Chern class of M.

The **(small) quantum cohomology** of M with coefficients in Λ_ω is defined by

$$QH^*(M) := H^*(M) \otimes_{\mathbb{Z}} \Lambda_\omega.$$

As before $H^*(M)$ denotes the ring $H^*(M;\mathbb{Z})$ modulo torsion. We now proceed to define the **quantum product** on $QH^*(M)$. We want the quantum product to be a linear homomorphism of Λ_ω-modules

$$QH^*(M) \otimes_{\Lambda_\omega} QH^*(M) \longrightarrow QH^*(M) : (a,b) \mapsto a * b.$$

Since $QH^*(M)$ is generated by the elements of $H^*(M)$ as a Λ_ω-module, it is enough to describe the multiplication for elements in $H^*(M)$. Let $e_0, e_1, \ldots, e_n$ be a basis for $H^*(M)$ (as a $\mathbb{Z}$-module). Assume each element is homogeneous and $e_0 = 1$, the identity for the usual product. Define the integer matrix

$$g_{ij} := \int_M e_i \smile e_j.$$

Here $e_i \smile e_j$ is the usual cup product in cohomology. Let g^{ij} be the inverse matrix. The quantum product of $a, b \in H^*(M)$, is defined by

$$a * b := \sum_{B \in H_2(M)} \sum_{k,j} \mathrm{GW}^M_{B,3}(a,b,e_k) g^{kj} e_j \otimes e^B. \tag{5}$$

The coefficients $\mathrm{GW}^M_{B,3}$ are the usual Gromov-Witten invariants of J-holomorphic curves in class B. The terms in the sum are nonzero only if $\deg(e_k) + \deg(e_j) = \dim M$ and $\deg(a) + \deg(b) + \deg(e_k) = \dim M + 2c_1(B)$. Thus, it is enough to consider classes B such that

$$\deg(a) + \deg(b) - \dim M \leq 2c_1(B) \leq \deg(a) + \deg(b).$$

In the problem at hand, a basis for $H^*(M)$ is given by the elements x_I as in 2.10. Then the integrals

$$g_{IJ} = \int_M x_I \smile x_J$$

all vanish unless the sets I and J are complementary. This is because if $I,\ J \subset \{1, \ldots, n\}$, $x_I \smile x_J = x_{\mathcal{S}}$ if and only if $I^c = J$. Here $x_{\mathcal{S}}$ is the positive generator of $H^{2n}(M;\mathbb{Z})$.

We claim that to compute the quantum product, we only need to consider in Equation (5) classes B such that $c_1(B) \geq 0$. More precisely, we have the proposition.

Proposition 3.1. *Assume (M,ω) is a symplectic manifold with a semi-free S^1-action with only isolated fixed points. Let $B \in H_2(M)$, and let $a, b, c \in H^*(M)$. If $c_1(B) < 0$, then the Gromov-Witten invariant $\mathrm{GW}^M_{B,3}(a,b,c)$ is zero. Moreover, if $c_1(B) = 0$ and some $\mathrm{GW}^M_{B,3} \neq 0$, then $B = 0$. Therefore, the expression for the quantum product (5) can be written as*

$$a * b = a \smile b + \sum_{B \in H_2(M), c_1(B) > 0} a_B \otimes e^B.$$

where the classes a_B have degree $\deg(a_B) = \deg a + \deg b - 2c_1(B)$.

Remark 3.2. *Note that since $c_1(B)$ is even, the classes a_B appear in the sum above by "jumps" of four in the degree.*

The rest of this section is dedicated to the proof of Proposition 3.1.

To compute the Gromov-Witten invariants $\mathrm{GW}^M_{B,3}(a,b,c)$ one usually constructs a regularization (virtual cycle) $\overline{\mathcal{M}}^{\nu}_{0,3}(M,J,B)$ of the moduli space $\overline{\mathcal{M}}_{0,3}(M,J,B)$. Then one computes the intersection number of the evaluation map

$$ev : \overline{\mathcal{M}}^{\nu}_{0,3}(M,J,B) \longrightarrow M^3$$

with a cycle $\alpha_1 \times \alpha_2 \times \alpha_3$ representing the class $\mathrm{PD}(a) \times \mathrm{PD}(b) \times \mathrm{PD}(c)$. This procedure can be modified in the following way. First, let $\alpha : Z \longrightarrow M^3$ be a pseudocycle that represents the product $\mathrm{PD}(a) \times \mathrm{PD}(b) \times \mathrm{PD}(c)$, then define the *cut-down* moduli space by

$$\overline{\mathcal{M}}_{0,3}(M,J,B;Z) := ev^{-1}(\overline{\alpha(Z)}).$$

Here $ev : \overline{\mathcal{M}}_{0,3}(M,J,B) \longrightarrow M^3$ is the evaluation map and $\overline{\alpha(Z)}$ is the closure in M of the pseudocycle Z [6]. Finally, construct a regularization of the cut-down moduli space. McDuff and Tolman use this approach to calculate the Gromov-Witten invariants. The next two results are proved in [4]. They show exactly how to compute the invariants $\mathrm{GW}^M_{B,3}$ using this procedure. Remember that an S^1 action on M can be extended to an action on J-holomorphic curves just by post-composition. Also, a pseudocycle $\alpha : Z \longrightarrow M$ is said to be S^1-invariant, if $\alpha(Z)$ is.

Proposition 3.3. *Let (M,ω) be a symplectic manifold. Then, the Gromov-Witten invariant $\mathrm{GW}^M_{B,3}(a,b,c)$ is a sum of contributions, one from each connected component of the moduli space $\overline{\mathcal{M}}_{0,3}(M,J,B;Z)$.*

Assume now that M is equipped with an S^1 action $\{\lambda_t\}$, and that $\alpha : Z \longrightarrow M^3$ and J are S^1-invariant. Then, a connected component of $\overline{\mathcal{M}}_{0,3}(M,J,B;Z)$ makes no contribution to $\mathrm{GW}^M_{B,3}(a,b,c)$ unless it contains an S^1-invariant element.

The following lemma describes what the invariant elements in the moduli space $\mathcal{M}_{0,k}(M,J,B)$ are. We include a proof so that Corollary 3.5 is a more natural result.

Lemma 3.4. *Let (M,ω) be a symplectic manifold with a semi-free S^1-action. Let $[u]$ be a class in the moduli space $\mathcal{M}_{0,k}(M,J,B)$ represented by a J-holomorphic sphere $u : \mathbb{P}^1 \longrightarrow M$. Assume $[u]$ is fixed by the action $\lambda = \{\lambda_\theta\}$. Then, there are at most two marked points, i.e. $k \leq 2$ and u can be parametrized as*

$$u : \mathbb{R} \times S^1 \longrightarrow M, \quad u(s,t) = \lambda_{pt}\gamma(s).$$

Here $\gamma : \mathbb{R} \longrightarrow M$ is a path joining two fixed points $x, y \in M$ so that the marked points are in $u^{-1}\{x,y\}$, and γ satisfies the gradient flow equation

$$\gamma'(s) = p \ \mathrm{grad}(H) \quad \textit{for some} \quad p \neq 0. \tag{6}$$

Moreover, if we fix γ, the parametrization is unique provided

$$x = \lim_{s \longrightarrow -\infty} \gamma(s) \ \textit{and}\ y = \lim_{s \longrightarrow \infty} \gamma(s).$$

Proof. Let $u : \mathbb{P}^1 \longrightarrow M$ be a non constant and not multiply covered J-holomorphic sphere in M. For each $\theta \in S^1$ the map $\lambda_\theta \circ u$ must be a reparametrization of u. This is because the equivalence class $[u]$ is fixed under the action. Thus, there is a $\phi_\theta \in \mathrm{PSL}(2,\mathbb{C})$ such that $\lambda_\theta \circ u = u \circ \phi_\theta$. Since the map u is not multiply covered ϕ_θ is unique. Then, it is easy to see that the assignment $S^1 \longrightarrow \mathrm{PSL}(2,\mathbb{C}) : \ \theta \mapsto \phi_\theta$ is a homomorphism. Since the only circle subgroups of $\mathrm{PSL}(2,\mathbb{C})$ are rotations about

an axis, we can choose coordinates on $\mathbb{P}^1$ so that the rotation axis is the line joining the unique fixed points $[0:1]$ and $[1:0]$. Assume that $\mathrm{Im}(u) \cap M^{S^1} = \{x, y\}$. Identify $\mathbb{P}^1/\{[0:1],[1:0]\}$ with the cylinder $\mathbb{R} \times S^1$ with complex structure j_0 defined by $j_0(\partial_s) = \partial_t$, $(s,t) \in \mathbb{R} \times S^1$. If $k = 2$ we identify the marked points $[0:1], [1:0]$ with the ends of the cylinder, so that $u([0:1]) = x$ and $u([1:0]) = y$. In general the image of the marked points must be fixed by the action. Therefore the marked points can be identify with a subset of $\{[0:1],[1:0]\}$. If $(s,t) \in \mathbb{R} \times S^1$ are the standard coordinates, then

$$\phi_\theta(s,t) = (s, t + q\theta), \text{ and } (\lambda_\theta \circ u)(s,t) = u(s, t+q\theta) \text{ where } q = \pm 1.$$

Define $\gamma(s) := u(s,0)$. Then we get $u(s,t) = \lambda_t \gamma(s)$. Since u is J-homomorphic and J is invariant

$$(\lambda_t)_*(\gamma'(s) + JX(\gamma(s))) = \partial_s u + J\partial_t u = 0.$$

With respect to the metric g_J, the gradient flow of H is given by $\mathrm{grad}H = -JX$, thus $\gamma'(s) = \mathrm{grad}(H)(\gamma(s))$. Now use that any sphere is a $|p|$-fold cover of a simple one. We absorb any negative sign into p rather than q. □

Note that our original goal was to understand the invariant stable maps in $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$. By the previous lemma, the non-constant components of the stable maps may carry at most two special points. Then the S^1-invariant elements in $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$ may have a *ghost* component that carries the third marked point.

We have as an immediate consequence the following corollary.

Corollary 3.5. *Assume the same hypothesis as in Lemma 3.4. Let u be an S^1-invariant sphere, and let $A \in H_2(M)$ be its homology class in M. Then its first Chern class is given by $c_1(A) = p(m(x) - m(y))$, and is always positive. Here $m(x)$ is the sum of the weights at x.*

Proof. Recall that $m(x) = n - \alpha(x)$ where $\alpha(x)$ is the Morse index of x. Now, if $m(x) < m(y)$, the path γ from x to y must satisfy Equation (6) with $p < 0$. This is because there are no generic solutions to this equation otherwise. Then $c_1(A) = p(m(x) - m(y))$. If $m(x) > m(y)$, now p must be positive and the result follows.

□

Remark 3.6. *Let u be an S^1-invariant holomorphic sphere, let $A \in H_2(M)$ be its homology class. Lemma 3.4 and Corollary 3.5 imply that if $c_1(A) = 0$ then A must be zero. This is because if A joins two fixed points $x, y \in M$, they must have the same index, which is not possible because the flow is assumed to be Morse-Smale.*

Proof of Proposition 3.1. By Proposition 3.3 a component of $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$ contributes to $\mathrm{GW}^M_{B,3}(a,b,c)$ only if the moduli space has a S^1-invariant stable map $\mathbf{u}$. We can assume that there is at least one non-trivial component u_i of the stable map $\mathbf{u}$. Since $\mathbf{u}$ is invariant, so is u_i. Therefore, Corollary 3.5 implies that $c_1(B_i) > 0$ if $B_i \in H_2(M)$ is the class of u_i. Then $c_1(B) > 0$ and the first claim follows. Note that the second part is a direct consequence Lemma 3.4, because any S^1-invariant J-holomorphic map with zero Chern class must be constant.

Finally , the product $a * b$ can be written as

$$a \smile b + \sum_{c_1(B)>0} \sum_I \mathrm{GW}^M_{B,3}(a, b, x_I) x_{I^c} \otimes e^B.$$

Now take

$$a_B := \sum_I \mathrm{GW}^M_{B,3}(a, b, x_I) x_{I^c}.$$

This proves the proposition. Note that we have $\deg(a_B) = \deg a + \deg b - 2c_1(B)$. □

3.2. **Almost Fano Manifolds.** Assume the hypothesis of Proposition 3.1. The relevant spheres (the ones that count for the GW invariants) all have positive first Chern class. Moreover, let $B \in H_2(M)$ be as in Proposition 3.1, then $c_1(B) > 0$. Since B is invariant, using Proposition 2.11 and Lemma 3.4, B can be written as a combination

$$B = \sum_i d_i p_i^-$$

where the coefficients d_i are non-negative integers. Therefore, if we define $A_i := p_i^-$ and $q_i := e^{A_i}$, we may now consider the polynomial ring

$$\Lambda = \mathbb{Q}[q_1, \dots, q_n]$$

as coefficients for the quantum cohomology. Then, if B is as before,

$$e^B = {q_1}^{d_1} \dots {q_n}^{d_n}.$$

This will be really useful in §4. For the rest of this paper, we will assume Λ to be the quantum coefficient ring.

We finish this section with a discussion about the behavior of J-holomorphic curves in M. In the literature an almost complex manifold (N, J) is said to be **Fano** if the first Chern class $c_1(TN, J)$ takes positive values on the **effective cone** $K^{\text{eff}}(N, J)$, namely

$$K^{\text{eff}}(N, J) := \{A \in H_2(N) | \exists \text{ a } J\text{-holomorphic curve in class } A\}.$$

In symplectic geometry sometimes is useful to consider the definition

$$K^{\text{eff}}(N, \omega) = \{A \in H_2(N) | A_1, \dots, A_n \in H_2(N) : A = \sum_i A_i, \mathrm{GW}^M_{A_i,3} \neq 0\}$$

for the effective cone on a symplectic manifold (N, ω, J) with a compatible almost complex structure J. Its clear that $K^{\text{eff}}(N, \omega) \subset K^{\text{eff}}(N, J)$. Then, we can say that (N, ω, J) is **almost Fano** if the first Chern class $c_1(TN, J)$ takes positive values on the effective cone $K^{\text{eff}}(N, \omega)$. We have the following corollary.

Corollary 3.7. *Let (M, ω) be a symplectic manifold with a semi-free S^1-action with isolated fixed points. Then (M, ω, J) is almost Fano.*

3.3. **The Seidel Automorphism.** In this paragraph we introduce the theory behind the definition of the Seidel element. The results concerning the present problem are discussed next. We will follow closely the book [6]. The proofs of the results exposed in this section are mostly contained in Chapters 8,9 and 11.

Let M be as in §1. Since the action is Hamiltonian, it is possible to associate to M the locally trivial bundle $\widetilde{M}_\lambda$ over $\mathbb{P}^1$ with fibre M defined by the *clutching function* (action) $\lambda : S^1 \longrightarrow \mathrm{Ham}(M, \omega)$:

$$\widetilde{M}_\lambda := S^3 \times_{S^1} M$$

We denote the fibres at $[1:0]$ and $[0:1]$ by M_0 and M_∞ respectively. Note that the isomorphism type of $\widetilde{M}_\lambda$ only depends on the homotopy class of λ.

Since λ is Hamiltonian, we can construct a symplectic form Ω on $\widetilde{M}_\lambda$. In fact the bundle $\pi : \widetilde{M}_\lambda \longrightarrow \mathbb{P}^1$ is a *Hamiltonian fibration* with fibre M, thus admitting sections ([6, Chapter 8]).

In the case when the manifold has an S^1-action, we choose an Ω-compatible almost complex structure $\widetilde{J}$ on $\widetilde{M}$, such that $\widetilde{J}$ is the product $J_0 \times J$ under trivializations. We can define for each fixed point $x \in M^{S^1}$ a pseudoholomorphic section $\sigma_x := \{[z_0 : z_1; x] : [z_0 : z_1] \in \mathbb{P}^1\}$.

Take $\widetilde{A} \in H_2(\widetilde{M}_\lambda, \Omega)$ a section class, that is $\pi_*(\widetilde{A}) = [\mathbb{P}^1]$. Let $a_1, a_2 \in H^*(M)$. Given two fixed marked points $w_1, w_2 \in \mathbb{P}^1$ we may think of the Poincaré dual to the class a_i as represented by a cycle Z_i in the fibre $M_i \hookrightarrow \widetilde{M}_\lambda$ over w_i. With this information it is possible to construct the Gromov-Witten invariant $\mathrm{GW}^{\widetilde{M}_\lambda, \mathbf{w}}_{\widetilde{A},2}(a_1, a_2)$. This invariant counts the number of J-holomorphic sections of $\widetilde{M}_\lambda$ in class $\widetilde{A}$ that pass through the cycles Z_i.

Definition 3.8. *Let (M, ω) be as before. Let $\sigma : \mathbb{P}^1 \longrightarrow \widetilde{M}_\lambda$ be a section. The* **Seidel automorphism**

$$\Psi(\lambda, \sigma) : QH^*(M; \Lambda) \longrightarrow QH^*(M; \Lambda)$$

is defined by

$$\Psi(\lambda, \sigma)(a) = \sum_{A \in H_2(M)} \sum_{k,j} \mathrm{GW}^{\widetilde{M}_\lambda, \mathbf{w}}_{[\sigma]+i_*A,2}(a, e_k) g^{kj} e_j \otimes e^A. \tag{7}$$

where $i : M \longrightarrow \widetilde{M}_\lambda$ is an embedding (as fibre).

In this definition we are considering a basis $\{e_i\}$ for $H^*(M)$ as in Equation (5). It is important to remark that the Seidel automorphism as defined above does not preserve degree. The shift on the degree depends on the section class σ that we use as reference.

If $\mathbb{1} \in QH^*(M)$ denotes the identity in the quantum cohomology ring, the class $\Psi(\lambda, \sigma)(\mathbb{1}) \in QH^*(M)$ is called the **Seidel Element** of the action respect to the section σ. We will use the same notation for the Seidel automorphism and the Seidel element. Thus, the Seidel automorphism is now given just by quantum multiplication by the element $\Psi(\lambda, \sigma)$ [6]. That is,

$$\Psi(\lambda, \sigma)(a) = \Psi(\lambda, \sigma) * a.$$

Note that the Seidel automorphism shifts degree by $\deg(\Psi(\lambda, \sigma))$.

3.4. **Seidel Automorphism and Isolated Fixed Points.** Consider now the present problem. That is, assume that the action is semi-free and it has isolated fixed points. Let $\sigma_{\max}$ be the section defined by the fixed point $p_{\mathcal{S}}$. In this particular case the automorphism $\Psi(\lambda, \sigma_{\max})$ increases the degree by $2n$. Let $p_I \in M$ be a fixed point. Recall that we can associate to p_I classes in homology p_I^- and p_I^+, and if we consider all the fixed points, then the classes p_I^+ form a basis for $H^*(M)$.

The next theorem, due to McDuff and Tolman [4], gives the first step towards a description of the Seidel automorphism. Although they have proved this result in great generality (the fixed points are allowed to be in submanifolds rather than being isolated) and they use quantum homology rather than cohomology, it is not hard to adapt their result to our present notation.

Theorem 3.9 (McDuff-Tolman)**.** *Let (M, ω) be a symplectic manifold with a semi-free circle action with isolated fixed points. Assume its associated Hamiltonian function H is such that $\int_M H\omega^n = 0$. Let $A_I \in H_2(M)$ be as considered in 2.11. Then, the Seidel automorphism can be expressed as*

$$\Psi(\lambda, \sigma_{\max})(\mathrm{PD}(p_I^-)) = \mathrm{PD}(p_I^+) \otimes e^{A_{I^c}} + \sum_{\omega(B)>0} a_B \otimes e^{A_{I^c}+B}.$$

where $a_B \in H^*(M)$. If $a_B \neq 0$ then $\deg x_I - \deg a_B = 2c_1(B)$. Moreover, if we write the sum above in terms of the basis $\{\mathrm{PD}(p_J^+)\}$ we get

$$\Psi(\lambda, \sigma_{\max})(\mathrm{PD}(p_I^-)) = \mathrm{PD}(p_I^+) \otimes e^{A_{I^c}} + \sum_{\omega(B)>0} \sum_{J\in\mathcal{S}} C_{B,J}\ \mathrm{PD}(p_J^+) \otimes e^{A_{I^c}+B}.$$

We know by Corollary 2.6 that $p_I^- = p_{I^c}^+$. By definition $\mathrm{PD}(p_J^+) = x_J$, therefore we have the following straightforward corollary.

Corollary 3.10. *Let (M, ω) be a symplectic manifold with a semi-free circle action with isolated fixed points. Assume its associated Hamiltonian function H is such that $\int_M H\omega^n = 0$. Let $\{x_I\}$ be the basis for the cohomology ring as considered in Remark 2.10, and let $A_I \in H_2(M)$ as considered in 2.11. The Seidel automorphism can be expressed as*

$$\Psi(\lambda, \sigma_{\max})(x_{I^c}) = x_I \otimes e^{A_{I^c}} + \sum_{\omega(B)>0} \sum_{J\in\mathcal{S}} C_{B,J}\ x_J \otimes e^{A_{I^c}+B}. \tag{8}$$

The rational coefficients $C_{\widetilde{B},J}$ can be nonzero only if $|I| - |J| = c_1(B)$ and the moduli space $\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; \overline{W^u(p_I)}, \overline{W^u(p_J)})$ has an S^1-invariant element. σ_I denotes the section defined by the fixed point p_I.

Thus, the key to understand the Seidel automorphism is first to know what the S^1-invariant elements in moduli spaces

$$\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; Z, Z')$$

are. Here Z and Z' are closed S^1-invariant cycles in M. These elements are called *invariant chains in section class $\sigma_z + A$ from $x \in Z$ to $y \in Z'$ with root z* [4]. We will explain what is the meaning of this.

Given $x, y, z \in M^{S^1}$ an invariant *principal* chain in section class $\sigma_z + A$ from $x \in Z$ to $y \in Z'$ with root z is a sequence of fixed points $x = x_1, \ldots, x_k = y$ joined by $\widetilde{J}$-holomorphic spheres with the following properties:

a) There is $1 \leq i_0 \leq k$ such that $x_{i_0} = x_{i_0+1} = z$, and they are joined by the section σ_z.
b) For each $1 \leq i < k$ where $i \neq i_0$, the points x_i, x_{i+1} are joined by an invariant sphere (in M) in class A_i.
c) $\sum_{i \neq i_0} A_i = A$.

An **invariant chain** in section class $\sigma_z + A$ from $x \in Z$ to $y \in Z'$ with root z is a chain as above with additional ghost components at each of which a tree of invariant spheres is attached. In this case, A is the sum of classes represented by the principal spheres and the bubbles.

Also, we can decompose $A = A' + A''$, where A' is the sum of spheres embedded in the fibre M_0 and A'' the ones in M_∞. An immediate lemma is the following

Lemma 3.11. *Assume the hypothesis of Corollary 3.10, and suppose $\sigma_z + A$ is an invariant chain in the moduli space*

$$\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; \overline{W^u(p_I)}, \overline{W^u(p_J)}).$$

Let $A = A' + A''$ be the decomposition of A as described above. Then, the first Chern classes $c_1(A'), c_1(A'')$ can be estimated by

$$c_1(A') \geq |m(x) - m(z)| \text{ and } c_1(A'') \geq |m(y) - m(z)|.$$

Therefore

$$\begin{aligned} c_1(A) &\geq |m(x) - m(z)| + |m(y) - m(z)|, \\ c_1(B) &\geq \max\{c_1(A'), c_1(A'')\}. \end{aligned} \tag{9}$$

Moreover if the coefficient $C_{B,J} \neq 0$, then $c_1(B) > 0$. Finally, observe that $c_1(A) = 0$ if and only if $A = 0$.

Proof. If A_i is an invariant sphere joining x_i to x_{i+1}, Lemma 3.4 shows that $c_1(A_i) \geq |m(x_i) - m(x_{i+1})|$. Then $c_1(A') \geq \sum_{i=0}^{k} |m(x_i) - m(x_{i+1})| \geq |m(x) - m(z)|$. The other part is analogous. Now, write $\sigma_I + B = A + \sigma_z$, since $x \in W^u(p_I)$, $m(x) > m(p_I)$, then $c_1(B) \geq c_1(A'')$. Similarly $c_1(B) \geq c_1(A')$. For the last statement, note that if $C_{B,J} \neq 0$ then $A \neq 0$. Then $A' \neq 0$ or $A'' \neq 0$. In any case $c_1(B) > 0$. For the last claim, note that if A_i is an invariant sphere with $c_1(A_i) = 0$, Remark 3.6 implies that A_i must vanish.

□

With Lemma 3.11 we can simplify the expression (8) to get the following corollary.

Corollary 3.12. *Assume the same hypothesis of Corollary 3.10. Then the Seidel element is given by*

$$\Psi(\lambda, \sigma_{\max})(x_{I^c}) = x_I \otimes e^{A_{I^c}} + \sum_{\omega(B)>0, c_1(A)>0} \sum_{J \in \mathcal{S}} C_{B,J}\ x_J \otimes e^{A_{I^c}+B}. \tag{10}$$

Again $C_{B,J} = 0$ unless $|I| - |J| = c_1(B)$ and the moduli space $\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; \overline{W^u(p_I)}, \overline{W^u(p_J)})$ has an S^1-invariant element.

Note that the only difference to Equation (8) is that we are considering only classes B with positive Chern number.

If there are any higher order terms, that is, terms that correspond to positive first Chern classes $c_1(B) > 0$, they contribute to the sum (10) as an element of degree

$2(|J| + c_1(A_{I^c} + B))$. Heuristically an invariant chain $A + \sigma_z$ makes a contribution only if $c_1(A)$ is big enough so that the inequalities (9) are satisfied. We will see in our next result that with our present hypotheses there are no such contributions. Thus there are not higher order terms. This result fails if for instance we allow the action to have fixed points along submanifolds, as we will see in the example described in §3.5. Observe that we can normalize our Hamiltonian function H (by adding a constant) so that $\int_M H\omega^n = 0$ without altering any of our previous results.

Theorem 3.13. *Let (M, ω) be a symplectic manifold with a semi-free circle action with isolated fixed points. Assume its associated Hamiltonian function H is such that $\int_M H\omega^n = 0$. Then, the Seidel automorphism $\Psi(\lambda, \sigma_{\max})$ acts on the basis $\{x_I\}$ by*

$$\Psi(\lambda, \sigma_{\max})(x_I) = x_{I^c} \otimes e^{A_I} \tag{11}$$

Proof. Consider I^c instead of I. By Corollary 3.12 the Seidel automorphism can be computed

$$\Psi(\lambda, \sigma_{\max})(x_{I^c}) = x_I \otimes e^{A_{I^c}} + \sum_{c_1(B)>0, J \in \mathcal{S}} C_{B,J} x_J \otimes e^{A_{I^c}+B}$$

As in Proposition 3.1, the Chern number $c_1(B)$ is a multiple of two. Thus the terms in the sum appear with "jumps" of four in the degree. By Corollary 3.12, $C_{B,J}$ is nonzero only if there is a S^1-invariant element in the moduli space $\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; \overline{W^u(p_I)}, \overline{W^u(p_J)})$. We want to see that the coefficients $C_{B,J}$ are all zero.

By contradiction assume there is an invariant chain $\sigma_z + A$ in this moduli space. Therefore A goes from a fixed point $x \in \overline{W^u(p_I)}$ to a fixed point $y \in \overline{W^u(p_J)}$. This chain satisfies

$$\sigma_z + A = \sigma_I + B. \tag{12}$$

Since the gradient flow is Morse-Smale and there is a gradient line from p_I to x, $m(x) \geq m(p_I) = n - 2|I|$. Analogously $m(y) \geq m(p_J) = n - 2|J|$. Since $c_1(B) = |I| - |J| > 0$ and we know $c_1(A) + m(z) = m(p_I) + c_1(B)$ from Equation (12), we get

$$c_1(A) = 2|K| - |I| - |J|, \tag{13}$$

where $K \subset \mathcal{S}$ is such that $p_K = z$.

Finally, from Lemma 3.11 we have

$$\begin{aligned} c_1(A) &\geq |m(x) - m(z)| + |m(y) - m(z)| \\ &\geq -2m(z) + m(y) + m(x) \\ &\geq 4|K| - 2|I| - 2|J|. \end{aligned}$$

Therefore, by Equation (13)

$$2|K| - |I| - |J| = c_1(A) \geq 2(2|K| - |I| - |J|).$$

This is only possible if $c_1(A) = 0$, i.e $2|K| - |J| = |I|$. By Lemma 3.11 A must be zero. Thus $x = y = z$. Therefore $B = \sigma_z - \sigma_I$. Hence $c_1(B) = m(z) - m(p_I) = 2(|I| - |K|)$. Since $c_1(A) = 0$, Equation (13) implies $|I| - |K| = |K| - |J|$. Thus $0 < c_1(B) = 2(|K| - |J|)$. By hypothesis $p_K = z = y \in \overline{W^u(p_J)}$. Then we have $|K| \leq |J|$. Thus $c_1(B) \leq 0$, which is a contradiction. This proves the theorem.

□

Corollary 3.14. *The Seidel element* $\Psi(\lambda, \sigma_{\max})$ *is given by*

$$\Psi(\lambda, \sigma_{\max}) = x_{\mathcal{S}}.$$

and the quantum product of $x_{\mathcal{S}}$ *with the element* x_I *is given by*

$$x_{\mathcal{S}} * x_I = x_{I^c} \otimes e^{A_I}. \tag{14}$$

Proof. The first part is obvious since $\Psi(\lambda, \sigma_{\max}) = \Psi(\lambda, \sigma_{\max}) * \mathbb{1} = \Psi(\lambda, \sigma_{\max}) * x_0 = x_{\mathcal{S}} \otimes e^0$. For the second part, observe that

$$x_{I^c} \otimes e^{A_I} = \Psi(\lambda, \sigma_{\max}) * x_I = x_{\mathcal{S}} * x_I.$$

□

The next paragraph is dedicated to discuss an example where the symplectic manifold has a semi-free circle action but the Seidel automorphism has higher order terms when evaluated on a particular class. In this example the fixed points are along submanifolds. This illustrates that we cannot have a result similar to Theorem 3.13 if we weaken one of our hypothesis.

3.5. **Example.** [4, Example 5.1] Let $M = \widetilde{\mathbb{P}^2}$ be the one point blow up of $\mathbb{P}^2$ with the symplectic form ω_μ so that on the exceptional divisor E, $0 < \omega_\mu(E) = \mu < 1$ and if $L = [\mathbb{P}^1]$ is the standard line, we have $\omega_\mu(L) = 1$. We can identify M with the space

$$\{(z_1, z_2) \in \mathbb{C}^2 | \ \mu \leq |z_1|^2 + |z_2|^2 \leq 1\}$$

where the boundaries are collapsed along the Hopf fibres. One of the collapsed boundaries is identified with the exceptional divisor. The other with L.

A basis for $H_*(M)$ is given by the class of a point pt, the exceptional divisor E, the fibre class $F = L - E$ and the fundamental class $[M]$. Note that the intersection products are given by $E \cdot E = -1$, $E \cdot F = 1$, $F \cdot F = 0$. Denote by b and f the Poincaré duals of E, F respectively. Then $b \cdot b = -1$ and $f \cdot f = 0$. It is not hard to see that the positive generator of $H^4(M)$ is $b \smile f = \mathrm{PD}(pt)$. Let us denote this class by just bf, so that a basis for the cohomology ring is $\{\mathbb{1}, b, f, bf\}$. Observe that M with the usual complex structure is Fano .

The non-vanishing Gromov-Witten invariants are given by

$$\mathrm{GW}^M_{L,3}(bf, bf, f) = \mathrm{GW}^M_{F,3}(bf, b, b) = 1;$$
$$\mathrm{GW}^M_{E,3}(c_1, c_2, c_3) = \pm 1 \text{ where } c_i = b \text{ or } f.$$

Let us consider the usual Novikov ring Λ_ω as the quantum coefficients. Then the quantum products are give by:

$$\begin{array}{lcllcl} bf * bf & = & (b+f) \otimes e^L & bf * f & = & \mathbb{1} \otimes e^L \\ bf * b & = & f \otimes e^F & b * b & = & -bf + b \otimes e^E + \mathbb{1} \otimes e^F \\ b * f & = & bf - b \otimes e^E & f * f & = & b \otimes e^E. \end{array}$$

In [4] it is proved that the circle action on M given by:

$$\alpha : (z_1, z_2) \mapsto (e^{-2\pi i t} z_1, e^{-2\pi i t} z_2), \qquad \text{for } 0 \leq t \leq 1.$$

is Hamiltonian. The maximum set of this action is exactly the points lying on the exceptional divisor E and the minimum set is the line L. After taking an appropriate reference section σ, the Seidel element $\Psi(\alpha, \sigma)$ is given by

$$\Psi(\alpha, \sigma) = b.$$

Thus, evaluating the Seidel map on the class f we have

$$\Psi(\alpha, \sigma)(f) = \Psi(\alpha, \sigma) * f = b * f = bf - b \otimes e^{E}.$$

Therefore the Seidel automorphism does have higher order terms when evaluated on the class f.

4. Proof of main result

Now we are ready for proving the main theorem. Recall that the quantum coefficient ring is $\Lambda = \mathbb{Q}[q_1, \dots, q_n]$. We also denote the usual cup product $a \smile b$ by ab for all $a, b \in H^*(M)$.

Proof of Theorem 1.1. This is an immediate consequence of the next lemma. $\square$

Lemma 4.1. *Let $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$, and let $1 \leq i \leq n$. Then*

$$x_{i_1} * \cdots * x_{i_k} = x_I \text{ and } x_i * x_i = \mathbb{1} \otimes e^{A_i} = q_i \tag{15}$$

Proof. To prove the first equality we will proceed by induction. Assume we have only two elements, say x_i, x_j, with $i \neq j$. Then, by Proposition 3.1 and Remark 3.2 we have

$$x_i * x_j = x_{\{ij\}} + c\ \mathbb{1} \otimes e^{B},$$

where the coefficient c is a rational number and $c_1(B) > 0$.

From Corollary 3.14 and the associativity of quantum multiplication we get

$$\begin{aligned} (x_{\mathcal{S}} * x_i) * x_j &= (x_{\{i\}^c} * x_j) \otimes e^{A_i} \\ &= x_{\{ij\}^c} \otimes e^{A_{ij}} + c\ x_{\mathcal{S}} \otimes e^{B}. \end{aligned} \tag{16}$$

By Proposition 3.1 the term $x_{\{i\}^c} * x_j$ is of the form

$$x_{\{i\}^c} x_j + \sum_{c_1(B')>0} a_{B'} \otimes e^{B'}$$

where again $\deg(a_{B'}) = \deg(x_{\{i\}^c}) + \deg(x_j) - 2c_1(B') < 2n$. Since $j \in \{i\}^c$, the term $x_{\{i\}^c} x_j$ is zero. Thus we have

$$\sum_{c_1(B')>0} a_{B'} \otimes e^{B'} \otimes e^{A_i} = x_{\{ij\}^c} \otimes e^{A_{ij}} + c\ x_{\mathcal{S}} \otimes e^{B}.$$

Then by comparing the degree of the coefficients in the previous equation, the constant c must vanish.

For the general case we will use the same argument. Assume the result holds for k different elements. Let $I' = \{i_{k+1}\} \cup I$. The quantum product $x_{i_1} * \cdots * x_{i_{k+1}}$ is by the inductive hypothesis, the same as $x_I * x_{i_{k+1}}$. This element can be written in terms of the basis as

$$x_I * x_{i_{k+1}} = x_{I'} + \sum_{c_1(B)>0, J \subset \mathcal{S}} a_{B,J}\ x_J \otimes e^{B}$$

where $2|J| = \deg(x_J) = \deg(x_{I'}) - 2d \leq \deg(x_{I'}) - 4$ and the coefficients $a_{B,J}$ are rational.

As before, using quantum associativity and Corollary 3.14 we get

$$(x_{\mathcal{S}} * x_I) * x_{i_{k+1}} = (x_{I^c} * x_{i_{k+1}}) \otimes e^{A_I} = x_{I'^c} \otimes e^{A_{I'}} + \sum_{c_1(B)>0, J\subset\mathcal{S}} a_{B,J}\ x_{J^c} \otimes e^{A_J+B}. \tag{17}$$

Here the degree satisfies

$$\deg(x_{J^c}) = 2n - \deg(x_{I'}) + 2d \geq 2n - \deg(x_{I'}) + 4 = 2(n - |I| + 1). \tag{18}$$

Now, the center term in Equation (17) is written as

$$(x_{I^c} x_{i_{k+1}} + \sum_{c_1(B')>0, K\subset\mathcal{S}} c_{B',K}\ x_K \otimes e^{B'}) \otimes e^{A_I},$$

where we have

$$\deg(x_K) \leq \deg(x_{I^c}) + \deg(x_{i_{k+1}}) - 4 = 2(n - |I| - 1). \tag{19}$$

Since $i_{i+1} \in I^c$, $x_{I^c} x_{i_{k+1}} = 0$. Finally we have the identity

$$\sum_{c_1(B')>0, K\subset\mathcal{S}} c_{B',K}\ x_K \otimes e^{B'+A_I} = x_{I'^c} \otimes e^{A_{I'}} + \sum_{c_1(B)>0, J\subset\mathcal{S}} a_{B,J}\ x_{J^c} \otimes e^{A_J+B}.$$

By Equations (18),(19), the coefficients $a_{B,J}$ are zero. This proves the first part of the lemma.

The second part is analogous, just write

$$x_i * x_i = x_i x_i + c\ \mathbb{1} \otimes e^B = c\ \mathbb{1} \otimes e^B$$

then multiplying by $x_{\mathcal{S}}$

$$(x_{\mathcal{S}} * x_i) * x_i = (x_{\{i\}^c} * x_i) \otimes e^{A_i} = c\ x_{\mathcal{S}} \otimes e^B.$$

Since $x_{\{i\}^c} * x_i = x_{\mathcal{S}}$, it follows that $c = 1$ and $e^B = e^{A_i}$.

□

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